# Introduction to Dynamical Systems

## Solutions Problem Set 7

**Exercise 1.** By considering the map $T\colon S^1 \longrightarrow S^1$ given by $T(z) = z^{10}$, show that for almost every $x \in (0,1]$ it holds that, writing

$$x = 0.x_1x_2\dots$$

in decimal expansion, we have that

$$\lim_{N\to\infty} \frac{1}{N} \left|\{j \in [1,2,\dots,N]\colon x_j = 0\}\right| = \frac{1}{10}.$$

*Solution.* We can apply the ergodic theorem to the function $\mathbb{1}_{(0,1)}$ and consider the map $T$ acting on $(0,10]$ periodically (which we relabel $\tilde{T}$) to find

$$\lim_{N\to\infty} \frac{1}{N} \left|\{j \in [1,2,\dots,N]\colon x_j = 0\}\right| = \frac{1}{10}\int_{(0,10]} \mathbb{1}_{(0,1)}(t)\,\mathrm{d}t = \frac{1}{10}, \quad \text{a.e. } x \in (0,1].$$

□

**Exercise 2.** Complete the proof of Lemma 1.2 in Lecture8.pdf, i.e. determine the constants $c, \theta_\pm$.

*Solution.* Recalling the notation from the proof, we have subspaces $\mathbb{R}^n = E_+ \oplus E_-$, such that by setting $A_\pm = A\big|_{E_\pm}$ to a hyperbolic matrix $A$, we can associate $\theta_\pm \in (0,1)$ and a constant $c \geq 0$ such that

$$\left|A_+^n(x)\right| \leq c\theta_+^n \left|x\right|,\, x \in E_+, \quad \left|A_-^{-n}(x)\right| \leq c\theta_-^n \left|x\right|,\, x \in E_-, \quad n \geq 0.$$

To give $\theta_\pm$ and $c$ explicitly, we recall the *spectral radius* $r_B$ of a matrix $B$ together with the formula,

$$r_B \coloneqq \max\left\{|\lambda| : \lambda \in \sigma(B)\right\}, \quad \lim_{n\to\infty} \left\|B^n\right\|^{1/n} = r_B,$$

where $\sigma(B)$ is the spectrum of $B$ and $\|\cdot\|$ is the operator norm. Given that the eigenvalues of $A_+$ all belong to $(0,1)$, we must have $r_{A_+} < 1$, and therefore for some $N \geq 0$, we have the estimates

$$\left\|A_+^n\right\|^{1/n} \leq \theta_+ < 1, \quad \forall n \geq N,$$

where $\theta_+$ can be taken arbitrarily close to $r_{A_+}$ depending only on $N$. Now, choosing

$$c = \max\left\{\frac{\left\|A_+^n\right\|}{\theta_+^n} : n \in \{1,\dots,N\}\right\},$$

we find that

$$\left\|A_+^n\right\|^{1/n} \leq c\theta_+, \quad \forall n < N$$

holds true, and we conclude that these are indeed the parameters in Lemma 1.2 thanks to the definition of the operator norm $\|\cdot\|$. The estimates for $\left|A_-^{-n}x\right|$ follow by applying the same argument to the linear map $A_-^{-1}$. □

**Exercise 3.** Let $A \in \mathrm{Mat}(n \times n; \mathbb{R})$ be a hyperbolic matrix, and let

$$\phi(x) = Ax + \hat{\phi}(x), \quad \psi(x) = Ax,$$

where $\hat{\phi}$ is in $C^1_b(\mathbb{R}^n, \mathbb{R}^n)$, i.e., bounded and continuously differentiable. Further assume that the Lipschitz constant for $\hat{\phi}$ is sufficiently small so that the conditions of Proposition 4.2 of Lecture8.pdf are satisfied. Then show that if $\phi(0) = 0$ and[1]

$$\text{trace}\left(D\hat{\phi}(0)\right) \neq 0,$$

the homeomorphism $h$ whose existence is asserted in Proposition 4.2 cannot be a $C^1$ diffeomorphism.

*Solution.* Since $h$ solves $\phi \circ h = h \circ \psi$, we have

$$A \cdot h + \hat{\phi} \circ h = h \circ A,$$

and by computing the Jacobian we reach

$$A \cdot Dh_x + D\hat{\phi}_{h(x)} \cdot Dh_x = Dh_x \cdot A$$

We then evaluate at $x = 0 = h(0)$ to get, assuming that h is a diffeomorphism, that

$$-A + Dh_0 \cdot A \cdot (Dh_0)^{-1} = D\hat{\phi}_0$$

But then we necessarily have

$$\text{trace}\left(D\hat{\phi}_0\right) = 0,$$

which contradicts the hypotheses. □

[1] Here $D\hat{\phi}$ is the Jacobian of the map